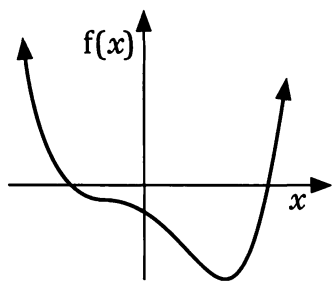
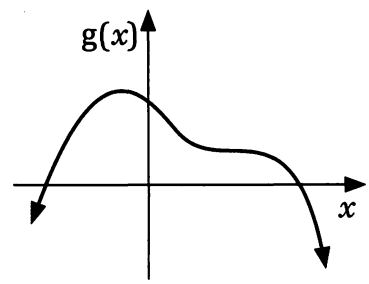
Chapter 2.

Applications of differentiation.

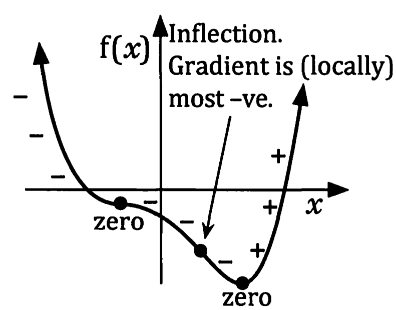
In your study of *Unit 2* of *Mathematics Methods* you would have seen how differentiation could be used to locate any stationary points on functions, and hence be useful in determining local maximum and local minimum values of functions. We will now see how the second derivative can be of use in this **optimisation** process.

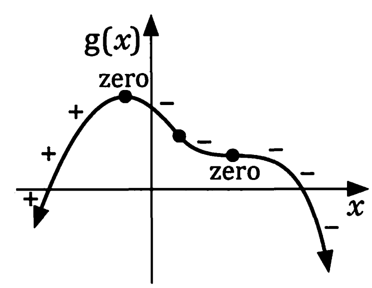
Examining the second derivative.
The diagrams below show the graphs of two functions f (x) and g (x).



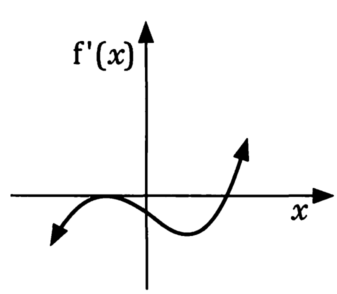


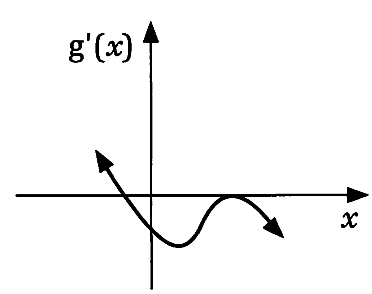
These functions are shown again below but now with the gradients at various places on the curves marked as positive, negative or zero.



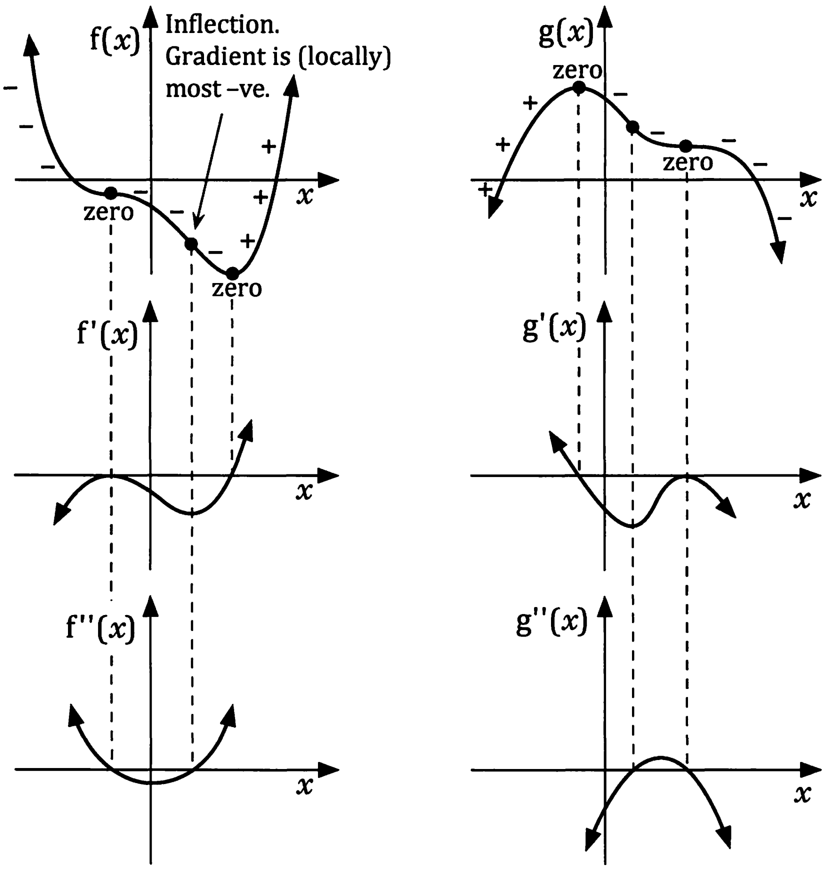


This allows a sketch of the first derivatives f '(x)
and g '(x) to be made:





The next page shows this process continued from f (x) and g (x), through f '(x) and g '(x) to f "(x) and g "(x).



Take particular notice of the following:

- Wherever $f''(x) < 0$ then $f(x)$ is concave down.
- Wherever $f''(x) > 0$ then $f(x)$ is concave up.
- At all of the points of inflection $f''(x)$ is zero.

Note: Care needs to be taken with the third dot point above. Whilst it is true that at all points of inflection the second derivative is zero, we cannot assume that if the second derivative is zero we necessarily have a point of inflection. Consider for example the function $y = x^4$. At the point (0, 0) the second derivative is zero, but on $y = x^4$ the point (0, 0) is a minimum point.

Locating turning points and points of inflection.

The properties of the second derivative stated above can be useful if we wish to determine the nature of any turning points on a curve, as the next example demonstrates.

The example also reminds us that the nature and location of any turning points can also be determined:
- by examining the sign of the gradient on either side of the turning point, (the sign test),

or
- from a calculator.

(Techniques you would already be familiar with from studying unit 2 of this course.)

Example 1
Clearly showing your use of calculus, determine the coordinates of any stationary points on the curve

$$y = x^3 - 12x^2 + 36x - 15$$

and state the nature of each.

If $\quad y = x^3 - 12x^2 + 36x - 15$

Then $\quad \frac{dy}{dx} = 3x^2 - 24x + 36$

For stationary points: $\quad \frac{dy}{dx} = 0,$

$$\begin{aligned}
\text{i.e.} \quad 3x^2 - 24x + 36 &= 0\\
\therefore \quad x^2 - 8x + 12 &= 0\\
(x-2)(x-6) &= 0\\
\text{Giving} \quad x &= 2 \quad \text{or} \quad 6\\
\text{When } x = 2 \quad y &= 2^3 - 12(2)^2 + 36(2) - 15\\
&= 17\\
\text{When } x = 6 \quad y &= 6^3 - 12(6)^2 + 36(6) - 15\\
&= -15
\end{aligned}$$

Thus $y = x^3 - 12x^2 + 36x - 15$ has two stationary points, one at (2, 17) and the other at (6, –15).

Determining the nature of the stationary points using the sign test:

Consider the sign of the gradient of the function either side of $x = 2$:

	$x = 1{\cdot}9$	$x = 2$	$x = 2{\cdot}1$
$\frac{dy}{dx} = 3(x-2)(x-6)$	+ve	zero	–ve
	/	—	\

Thus (2, 17) is the local maximum. Similar working shows (6, –15) is a local minimum.

Determining the nature of the stationary points using the second derivative test:

With $\frac{dy}{dx} = 3x^2 - 24x + 36$ it follows that $\frac{d^2y}{dx^2} = 6x - 24$.

If $x = 2$

$\frac{d^2y}{dx^2} = -\text{ve}$

$\therefore$ concave down, a maximum.

If $x = 6$

$\frac{d^2y}{dx^2} = +\text{ve}$

$\therefore$ concave up, a minimum.

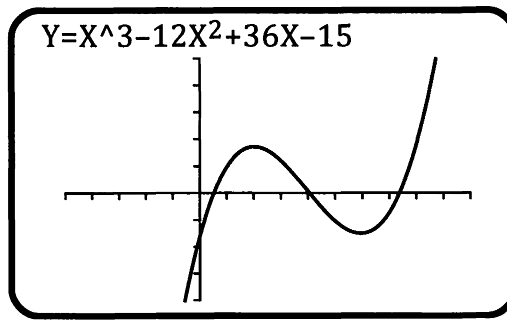
Thus, as before, (2, 17) is a local maximum and (6, –15) is a local minimum.

Note

- The second derivative test uses the facts stated earlier, i.e.:
  - ☞ if $f''(x) < 0$, i.e. negative, then $f(x)$ is concave down.
    Hence **with $f'(x) = 0$ and $f''(x)$ negative we have a maximum** point.
  - ☞ if $f''(x) > 0$, i.e. positive, then $f(x)$ is concave up.
    Hence **with $f'(x) = 0$ and $f''(x)$ positive we have a minimum** point.

- **If $f'(x) = 0$ and $f''(x) = 0$ then we could have maximum, minimum or inflection and would need to investigate further using the sign test.**

- The nature of each turning point could alternatively have been determined by viewing a graphical display of the function. From this it can be seen that the turning point at (2, 17) is a local maximum and the one at (6, –15) is a local minimum.



- The whole task could be completed using a calculator but some explanation and method would need to be shown to ensure that you met the requirement to *clearly show your use of calculus.*

Define $f(x)=x^3-12\cdot x^2+36x-15$

done

$\text{solve}\left(\frac{d}{dx}(f(x)) = 0, x\right)$

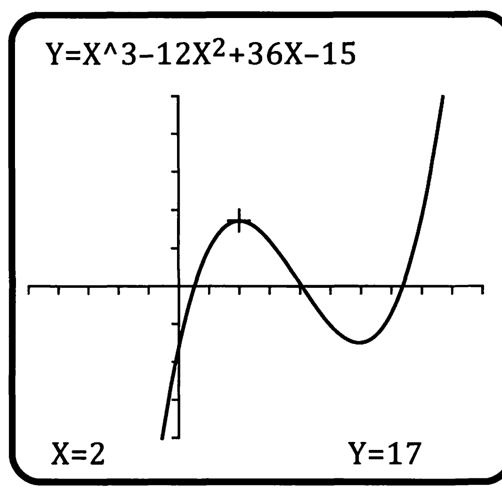
$\{x = 2,\ x = 6\}$

f(2)

17

f(6)

-15

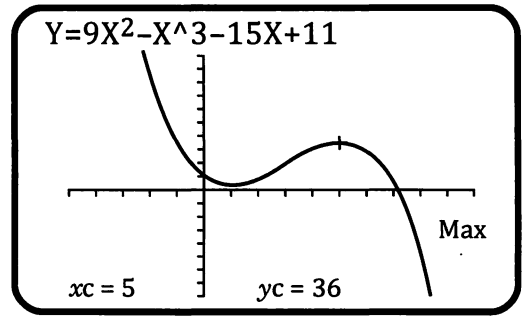


- What we found in the last example was the **local** maximum (and **local** minimum) i.e. the point which is a maximum point compared to others in that locality. In some cases we may be concerned with the maximum or minimum value a function can take for some interval $a \le x \le b$. We are then concerned with the **global** maxima, which may or may not coincide with the local maxima.

For example consider the graph of

$$f(x) = 9x^2 - x^3 - 15x + 11$$

with the local maximum at (5, 36), as shown on the right.
If we were asked for the maximum value of this function in the interval, $-2 \le x \le 7$ we see that f(−2) will give this greatest value. With f(−2) = 85 we say that the global maximum for the interval $-2 \le x \le 7$ is 85.



Example 2

Clearly showing your use of calculus, determine the <u>exact</u> coordinates of any stationary points on the curve $y = 2x + \frac{6}{x}$ and use the second derivative test to determine the nature of each.

If $\quad y = 2x + \frac{6}{x} \quad \left(= 2x + 6x^{-1}\right) \quad$ then $\quad \frac{dy}{dx} = 2 - \frac{6}{x^2}$

Stationary points will occur where $\quad \frac{dy}{dx} = 0, \quad$ i.e. $\quad 2 - \frac{6}{x^2} = 0$

$$\therefore \quad x^2 = 3$$

Giving $\quad x = \sqrt{3} \quad$ or $\quad -\sqrt{3}$

Thus there are two stationary points, when $x = \sqrt{3}$ and when $x = -\sqrt{3}$.

When $x = \sqrt{3}$ $\quad y = 2\sqrt{3} + \frac{6}{\sqrt{3}} = 2\sqrt{3} + \frac{6}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$

$$= 2\sqrt{3} + 2\sqrt{3}$$

$$= 4\sqrt{3}$$

Thus a stationary point exists at $(\sqrt{3}, 4\sqrt{3})$.

Similarly, when $x = -\sqrt{3}$, $y = -4\sqrt{3}$. The other stationary point is at $(-\sqrt{3}, -4\sqrt{3})$.

Second derivative test.

With $\quad \frac{dy}{dx} = 2 - \frac{6}{x^2} \quad \left(= 2 - 6x^{-2}\right) \quad$ it follows that $\quad \frac{d^2y}{dx^2} = \frac{12}{x^3}$.

If $\quad x = \sqrt{3}$	If $\quad x = -\sqrt{3}$
$\frac{d^2y}{dx^2} = +\text{ve}$	$\frac{d^2y}{dx^2} = -\text{ve}$
∴ minimum	∴ maximum

The point $(\sqrt{3}, 4\sqrt{3})$ is a local minimum and $(-\sqrt{3}, -4\sqrt{3})$ a local maximum.

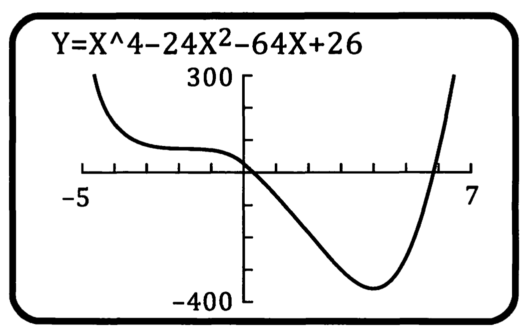
Sketching graphs.

The ability to determine the location and nature of any turning points on a graph, together with an ability to determine any intercepts with the axes, asymptotes, any symmetry that may exist and the behaviour of the function as $x \to \pm\infty$, all help in the production of a sketch of the graph of a function.

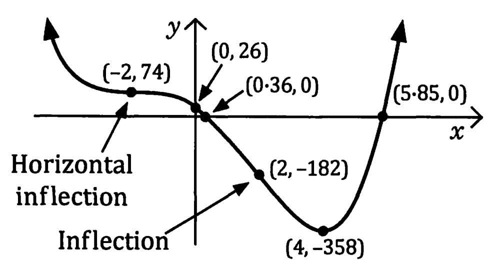
☞ Whilst we would not expect to read values from a sketch graph with any great accuracy the sketch should be neatly drawn and should show the noteworthy features of the graph.

☞ The reader should already be familiar with sketching graphs from studying unit two of this course. However, we now have the second derivative test that we can use and the ability to differentiate more complicated functions.

Example 3 (Sketching <u>with</u> the assistance of a graphic calculator.)
With the aid of a graphic calculator produce a sketch of $y = x^4 - 24x^2 - 64x + 26$ indicating on your sketch the location of any stationary points, intercepts with the axes and points of inflection that appear in the interval $-5 \le x \le 7$. (If any rounding is necessary give answers correct to 2 decimal places.)



By substituting $x = 0$ we can determine that the curve cuts the y-axis at $(0, 26)$.
Displaying the graph (see diagram on the right) indicates that the curve cuts the x-axis at approximately $(0{\cdot}5, 0)$ and $(6, 0)$, has a minimum point at or near $x = 4$ changes concavity at or near $x = -2$ (horizontal(?) inflection) and at or near $x = 2$ (inflection).
Using the calculator facilities to determine the coordinates of these points allows a sketch to be made:



Example 4 (Sketching <u>without</u> the assistance of a graphic calculator.)
For the function $y = x^3 + 3x^2 - 24x + 20$ and <u>without</u> the assistance of a calculator determine:

(a) The coordinates of the y-axis intercept.

(b) The behaviour of the function as $x \to \pm\infty$.

(c) The location and nature of any turning points.

(d) The coordinates of any points for which $\dfrac{d^2y}{dx^2} = 0$.

(e) Show your answers to the previous parts on a sketch of the graph.

(a) $y = x^3 + 3x^2 - 24x + 20$

When $x = 0$ $y = 20$

The y-axis intercept has coordinates (0, 20).

(b) For $x \to \pm\infty$ the x^3 term will dominate.

Thus as $x \to +\infty$ $y \to +\infty$ (and faster than x does).

and as $x \to -\infty$ $y \to -\infty$ (and faster than x does).

(c) With
$$y = x^3 + 3x^2 - 24x + 20$$
$$\frac{dy}{dx} = 3x^2 + 6x - 24 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x + 6$$
$$= 3(x^2 + 2x - 8)$$
$$= 3(x + 4)(x - 2)$$

Thus when $x = -4$ $y' = 0$ and $y'' = -\text{ve}$. A maximum point.

For this value of x
$$y = (-4)^3 + 3(-4)^2 - 24(-4) + 20$$
$$= -64 + 48 + 96 + 20$$
$$= 100$$

The function has a maximum turning point at (−4, 100).

Also when $x = 2$ $y' = 0$ and $y'' = +\text{ve}$. A minimum point.

For this value of x
$$y = (2)^3 + 3(2)^2 - 24(2) + 20$$
$$= 8 + 12 - 48 + 20$$
$$= -8$$

The function has a minimum turning point at (2, −8).

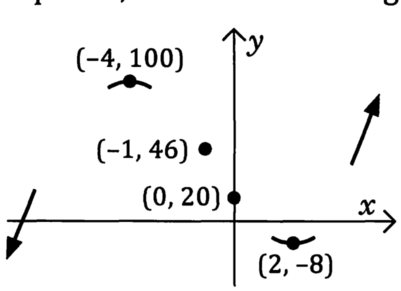
(d)
$$\frac{d^2y}{dx^2} = 0 \quad \text{when} \quad 6x + 6 = 0$$

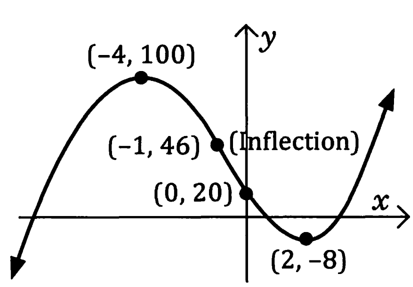
i.e. when $x = -1$

For this value of x
$$y = (-1)^3 + 3(-1)^2 - 24(-1) + 20$$
$$= -1 + 3 + 24 + 20$$
$$= 46.$$

The second derivative is zero at the point (−1, 46).

(e) Placing the above information on a graph, below left, a sketch of the function can be completed, as shown below right.

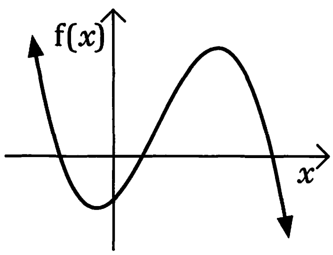




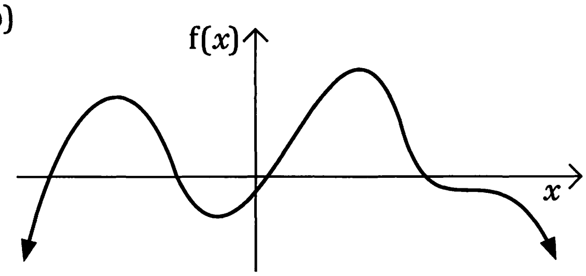
Exercise 2A

1. Copy the following graphs and then draw $f'(x)$ and $f''(x)$ for each one.

(a)



(b)



Use calculus techniques to determine the exact coordinates of any stationary points on the following curves, and use the second derivative test (and the sign test if necessary) to determine whether maximum, minimum or horizontal inflection.

2. $y = x^2 - 12x + 40$

3. $y = 5 + 8x - x^2$

4. $y = x^3 - 9x$

5. $y = x^3 - 9x^2 - 21x + 60.$

6. $y = (x - 1)^4 + 2$

7. $y = x + \dfrac{4}{x+3}$

8. $y = x + \dfrac{5}{x}$

9. $y = (2x - 1)^5 + 1$

10. With the aid of a graphic calculator produce a sketch of
$$y = x^3 - 9x^2 + 15x + 74$$
indicating on your sketch the location of any stationary points, intercepts with the axes and points of inflection.

11. For the function $y = x^3 - 6x^2 - 15x + 30$ and <u>without</u> the assistance of a calculator determine:
    (a) The coordinates of the y-axis intercept.
    (b) The behaviour of the function as $x \to \pm\infty$.
    (c) The location and nature of any turning points.
    (d) The coordinates of any points for which $\dfrac{d^2y}{dx^2} = 0$.
    (e) Show your answers to the previous parts on a sketch of the graph of the function.

12. For the function $y = x^4 - 4x^3 + 1$ and <u>without</u> the assistance of a calculator determine:
    (a) The coordinates of the y-axis intercept.
    (b) The behaviour of the function as $x \to \pm\infty$.
    (c) The location and nature of any turning points.
    (d) The coordinates of any points for which $\frac{d^2y}{dx^2} = 0$.
    (e) Show your answers to the previous parts on a sketch of the graph of the function.

13. Use the ability of your calculator to differentiate algebraic expressions to determine $\frac{dy}{dx}$ given that $y = (x - 3)^3(3x + 7)$, giving your answer in factorised form.
    See if you can obtain this same factorised answer using the product rule and chain rule and without the assistance of your calculator.
    Without the assistance of your calculator, determine the coordinates of any stationary points on the curve
    $$y = (x - 3)^3(3x + 7)$$
    and use the sign test and/or the second derivative to determine the nature of each.

14. The graph of $$f(x) = \frac{x^3}{8} - x^2 + 2x + 1$$
    has a local maximum at $(\frac{4}{3}, 2\frac{5}{27})$ and a local minimum at $(4, 1)$.
    Using your calculator purely to assist with the arithmetic, if necessary, determine the maximum value of $f(x)$ for (a) $0 \le x \le 5$,
    (b) $0 \le x \le 6$.

For questions 15 to 17

(a) use calculus to determine the coordinates of any points where $f''(x) = 0$,

(b) by checking a graphic calculator display of the graph of $f(x)$ state whether each point from (a) is a point of inflection or not. If yes, determine whether it is horizontal inflection.

15. $f(x) = x^3 - 12x$

16. $f(x) = 8x^3 - x^4$

17. $f(x) = x^4$

18. For $f(x) = x^3 - 3x - 2$ determine $f'(x)$,
    $f''(x)$,
    and find a such that $f''(a) = 0$.
    By considering the sign of $f''(x)$ on either side of $x = a$, to see if the concavity changes, state whether $(a, f(a))$ is a point of inflection or not.

Rates of change.

Given y in terms of x the process of differentiation gives us $\frac{dy}{dx}$, the rate of change of y with respect to x.

If instead of y we use V, where V represents volume, and in place of x we use t, where t represents time, we can use differentiation to determine $\frac{dV}{dt}$, the rate of change of volume with respect to time.

Similarly if we are told a rule relating A m^2, the area of a particular algal bloom, to T, the temperature of the surroundings in degrees Celsius, then differentiation can be used to determine $\frac{dA}{dT}$, the rate of change of A with respect to T, in m^2/°C.

Example 5

A colony of bacteria is increasing in such a way that the number of bacteria present after t hours is given by N where $N = 250 + 100t + 50t^3$.

Find (a) the number of bacteria present when $t = 5$,

(b) an expression for the instantaneous rate of change of N with respect to t,

(c) the rate the colony is increasing, in bacteria/hour, when (i) $t = 2$, (ii) $t = 10$.

(a)

$$N = 250 + 100t + 50t^3$$

When $t = 5$

$$N = 250 + 100(5) + 50(5)^3 = 7000$$

When $t = 5$ there are 7000 bacteria present.

(b)

$$N = 250 + 100t + 50t^3$$

$$\therefore \quad \frac{dN}{dt} = 100 + 150t^2$$

The instantaneous rate of change of N with respect to t is

$(100 + 150t^2)$ bacteria/hr.

(c) (i)

$$\frac{dN}{dt} = 100 + 150t^2$$

when $t = 2$ $\quad \frac{dN}{dt} = 700$

When $t = 2$, the colony is increasing at 700 bacteria/hr.

(ii) When $t = 10$ $\quad \frac{dN}{dt} = 15100$

When $t = 10$, the colony is increasing at 15100 bacteria/hr.

Define $N(t)=250+100t+50t^3$	done
$N(5)$	7000
$\frac{d}{dt}(N(t))$	$150t^2+100$
$\frac{d}{dt}(N(t))\|t=2$	700
$\frac{d}{dt}(N(t))\|t=10$	15100

Exercise 2B

1. If $P = 2a^3 + 3a - 7$ find an expression for the rate of change of P with respect to a.

2. If $Y = p - 5p^2 + 2p^3$ find an expression for the rate of change of Y with respect to p.

3. If $Q = (2t - 1)^3$ find an expression for the rate of change of Q with respect to t.

4. If $A = \dfrac{3x - 2}{2x + 5}$ find an expression for the rate of change of A with respect to x.

5. If $P = (2q - 5)(3q^2 + 1)$ find an expression for the rate of change of P with respect to q.

6. If $V = (1 + 0{\cdot}5t)^3$ find the rate of change of V (cm^3) with respect to t (seconds) when
   (a) $t = 2$, (b) $t = 6$, (c) $t = 10$.

7. A colony of flying insects takes over a nesting site and the population of the colony grows such that N, the number of insects present, t days after taking over the site, is given by $N = 500 - 5t^2 + 10t^3$.
   Find (a) an expression for the rate of change of N with respect to t,
   (b) the rate at which the population is changing when
   (i) $t = 1$, (ii) $t = 5$, (iii) $t = 10$.

8. For the first 20 seconds of its motion the height of a particular rocket above the earth's surface, t seconds after launch, is h metres where $h = 5t(1 + 2t)$.
   Find both the height and the rate of change of height with respect to time when
   (a) $t = 1$, (b) $t = 5$, (c) $t = 20$.

9. A colony of bacteria is increasing in such a way that the number of bacteria present after t hours is given by N where $N = 5(2t + 1)^3$.
   Find (a) the number of bacteria present initially (i.e. when $t = 0$),
   (b) the number of bacteria present when $t = 5$,
   (c) an expression for the instantaneous rate of change of N with respect to t,
   (d) the rate the colony is increasing, in bacteria/hour, when
   (i) $t = 2$, (ii) $t = 5$, (iii) $t = 10$.

10. A charity group launches a new appeal to raise \$15 000 for a particular project. A computer analysis of previous appeals indicates that for this appeal, \$$R$, the amount by which the funds raised fall below the target of \$15 000, and w, the number of weeks after the much publicised launch, will approximately follow the rule

$$R = 15\,000 - 5000\sqrt{w} - \frac{800}{w + 1}$$

According to this model
(a) How many weeks after the launch could the organisers expect the target to be reached? (Hint: Use a graphic calculator.)
(b) Find an expression for the rate of change of R with respect to w and evaluate this expression (to nearest \$100) for $w = 1$, $w = 3$ and $w = 8$.

Acceleration.

The *Preliminary Work* section at the beginning of this book reminded us that one of the commonest rates of change that concerns us is the rate at which we change our location. If we measure our location as a **displacement** from some fixed point or origin, then the rate at which we change our displacement is our **velocity**.

Thus if the displacement is x [= f(t)] then v, the velocity as a function of time, is given by:

$$v = \frac{dx}{dt}$$

Similarly the rate of change of velocity with respect to time, $\frac{dv}{dt}$, gives **acceleration,** a.

$$\text{Thus} \qquad a = \frac{dv}{dt}$$
$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d^2x}{dt^2}.$$

For example, if $\quad x = 5t^3 + 6t^2 + 7t + 1$

then $\quad v = \frac{dx}{dt} = 15t^2 + 12t + 7$

and $\quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 30t + 12.$

Example 6

A body moves in a straight line such that its displacement from an origin O, at time t seconds, is x metres where $x = t^3 + 6t + 5$.

Find the acceleration of the body when $t = 3$.

If $\quad x = t^3 + 6t + 5$

then $\quad v = \frac{dx}{dt} = 3t^2 + 6$

and $\quad a = \frac{dv}{dt} = 6t.$

Thus, when $t = 3 \quad a = 6(3)$

When $t = 3$ the acceleration is $18\,\text{m/s}^2$.

$\frac{d^2}{dt^2}(t^3+6t+5)\,|\,t=3$

18

Example 7
A body moves in a straight line such that its displacement from an origin O, at time t seconds, is x metres where $x = 5t^2 + 7t + 3$.
Find the acceleration of the body when $t = 0$ (i.e. the *initial* acceleration of the body).

If $\quad x = 5t^2 + 7t + 3$

then $\quad v = \dfrac{dx}{dt} = 10t + 7$

and $\quad a = \dfrac{dv}{dt} = 10.$

(I.e., the acceleration is a constant 10 m/s^2.)

Thus the initial acceleration is 10 m/s^2.

$$\boxed{\frac{d^2}{dt^2}(5t^2+7t+3)\Big|t=0 \qquad 10}$$

Note: • We have already seen that we write y' for $\dfrac{dy}{dx}$ and y'' for $\dfrac{d^2y}{dx^2}$.

For differentiation with respect to time we tend to use a dot notation rather than a dash. Thus $\dfrac{dy}{dt}$ is sometimes written as $\dot{y}$ and $\dfrac{d^2y}{dt^2}$ is sometimes written as $\ddot{y}$.

• The relationship between displacement, velocity and acceleration is shown below:

Displacement from O, x metres. → Differentiate with respect to time, t seconds. → Velocity, v m/s → Differentiate with respect to time, t seconds. → Acceleration, a m/s^2

(Different units of displacement and time, eg km and hours, would give different units for velocity, km/hr, and acceleration, km/hr^2.)

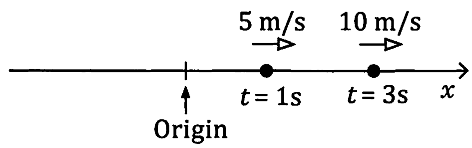
• Displacement, velocity and acceleration are **vector** quantities, i.e. they have size and direction. This section will only consider rectilinear motion, i.e. motion in a straight line. For such motion there are only two directions possible and these are distinguished by use of positive and negative.

• If we take positive as being "to the right", a body with positive acceleration will either be moving to the right and increasing its speed or moving to the left and decreasing its speed.

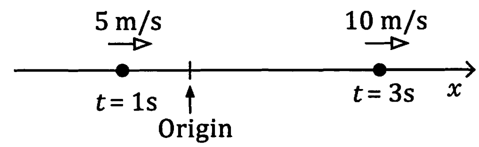
Exercise 2C

1. Each of the following diagrams show the velocity and location of a body at two times. Given that the body is experiencing constant acceleration state whether this acceleration is positive or negative.

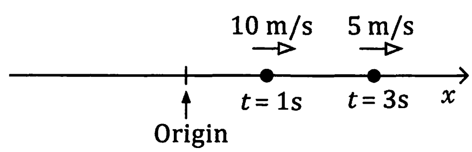
(a)



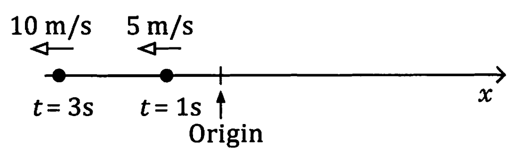
(b)



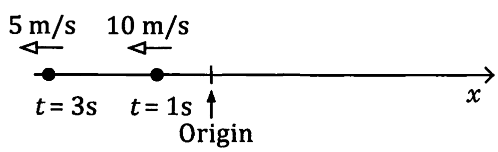
(c)



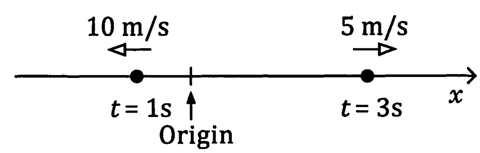
(d)



(e)



(f)



Questions 2 and 3 involve a body moving in a straight line with its displacement, x metres, given as a function of the time, t seconds.

2. If $x = 5t^2 + 6t$ find
   (a) the velocity when $t = 2$,
   (b) the acceleration when $t = 3$.

3. If $x = \dfrac{(2t+1)^3}{10}$ find
   (a) the velocity when $t = 2$,
   (b) the acceleration when $t = 2$.

Questions 4 and 5 involve a body moving in a straight line with its velocity, v metres/second, given as a function of the time, t seconds.

4. If $v = \dfrac{2t+3}{t+1}$ find the acceleration when $t = 4$.

5. If $v = (2t-1)^5$ find the acceleration when $t = 1{\cdot}5$.

6. For each of the following, x metres is the displacement of a body from an origin O, at time t seconds. Find the instantaneous acceleration of the body for the given value of t.

   (a) $x = 2t^3 + 4,\ \ t = 2.$ (b) $x = 7t,\ \ t = 3.$

   (c) $x = \dfrac{27}{2t+1},\ \ t = 1.$ (d) $x = \sqrt{2t+1},\ \ t = 4.$

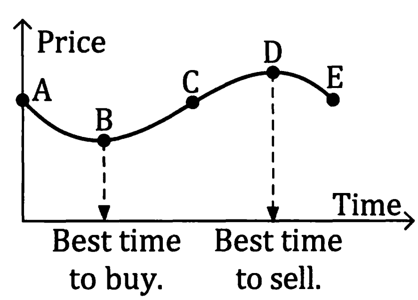
   (e) $x = (9 - 2t)^4,\ \ t = 4.$ (f) $x = 2t\,(1 + 5t)^3,\ \ t = 0{\cdot}4.$

7. The displacement of a body from an origin O, at time t seconds, is x metres where
$$x = t^2 - 11t + 3,\ \ t \geq 0.$$
   (a) Find the initial velocity of the body.
   (b) Find the acceleration of the body.
   (c) The value of t for which the body has a velocity of 5 m/s.
   (d) The values of t for which the body has a speed of 5 m/s.
   Note: To have a *speed* of 5 m/s the velocity can be 5 m/s or –5 m/s.

8. The displacement of a body from an origin O, at time t seconds, is x metres where
$$x = 27t + 3t^2 - \frac{t^3}{3} - 90,\ \ t \geq 0.$$
   Find the displacement of the body from O when the acceleration is zero.

Optimisation.

Let us suppose that the selling price of housing units in a large block of such units varies over a period of time in the manner indicated in the diagram on the right.
For someone buying a unit the most desirable or **optimum** time to buy will be at the time given by point B. This is the time during the period under consideration when the price is at its lowest.
For someone selling a unit the most desirable or optimum time to sell will be at the time given by point D. This is the time during the period under consideration when the price is at its highest.



The techniques used earlier in this chapter to determine the location and nature of any maximum or minimum points on a curve can be used to determine the most desirable or optimum situation for an event to occur. As the *Preliminary Work* section at the beginning of this text stated, this application of differentiation to determine an optimal solution should not be new to you. However now, from the earlier work of this chapter, you also have the **second derivative test** available to use when attempting to determine the nature of any stationary points on the function involved.

The following list should serve to remind you of the steps to follow when locating stationary points to solve applied optimisation problems.

① If a diagram is not given then draw one if it helps.

② Identify the variable that is to be maximised, or minimised. If this variable is, say, C then you must find an equation with C as the subject. i.e. C = ???.

③ If this equation for C involves two variables (other than C) find another equation that will allow us to substitute for one of the variables.

④ When you have C in terms of one variable, say x, then you could view the function on your calculator and locate any turning points, or, if the use of calculus is to be demonstrated, find the values of x for which $\frac{dC}{dx} = 0$.

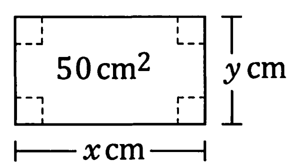
⑤ Use the second derivative or the sign test or a graphic calculator display to determine whether maximum or minimum.

⑥ Check that the value of x for the required maximum, or minimum, is within the values that the situation allows x to lie and check that it gives the global maximum, or minimum.

Example 8

The area of a rectangle is to be 50 cm^2. Find the dimensions of the rectangle if its perimeter is to be a minimum.

With the perimeter of the rectangle as P cm and x and y as shown in the diagram on the right, it follow that



$$\begin{array}{lrcllrcl} & xy & = & 50 & \text{and} & \mathrm{P} & = & 2x + 2y \\ \text{Thus} & y & = & \dfrac{50}{x} & \text{and so} & \mathrm{P} & = & 2x + \dfrac{100}{x} \\ & & & & & & = & 2x + 100x^{-1} \\ & & & & \therefore & \dfrac{d\mathrm{P}}{dx} & = & 2 - 100x^{-2} \\ & & & & & & = & 2 - \dfrac{100}{x^2} \\ & & & & \text{and} & \dfrac{d^2\mathrm{P}}{dx^2} & = & 200x^{-3} \\ & & & & & & = & \dfrac{200}{x^3} \end{array}$$

If $\frac{d\mathrm{P}}{dx} = 0$ then $2 - \frac{100}{x^2} = 0$ giving $x = \sqrt{50}$ $\quad$ ($-\sqrt{50}$ not applicable).

$$= 5\sqrt{2}.$$

For this value of x, $\frac{d^2\mathrm{P}}{dx^2}$ is positive and $y = 5\sqrt{2}$.

Hence the dimensions for the perimeter to be a minimum are $5\sqrt{2}$ cm $\times$ $5\sqrt{2}$ cm.

Exercise 2D

For each of the following questions use your calculator when appropriate but do make sure that you demonstrate **your** use and understanding of the calculus processes involved and, in particular, your use of the second derivative test when appropriate.

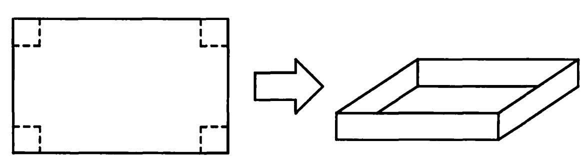
1. The total profit, \$$P$, generated from the production and marketing of x items of a certain product is given by $P = 25x^2 + 5000x - x^3$.
   Find the value of x that gives maximum profit and determine what this maximum profit will be.

2. The total profit, \$$P$, generated from the production and marketing of x items of a certain product is given by $P = 10\,000x - x^3 + 275x^2 - 1\,000\,000$.
   How many items should be made for maximum profit?
   What would this maximum profit will be?

3. The profit, \$P, made by a company producing and marketing x items of a certain product is given by: $\mathrm{P} = -\frac{x^3}{3} + 20x^2 + 2\,100x - 25\,000$.
   Use calculus methods, and showing full reasoning, find the value of x for maximum profit and determine this maximum profit (nearest \$1 000).

4. Let us suppose that for an orchard involving a particular type of fruit tree, the average weight of fruit, w kg, produced per tree in a year depends on N, the number of trees per 100 m^2, according to the rule w = (600 – 15N), for $10 \leq N \leq 25$.
   Using calculus, and clearly justifying that your value would indeed give a maximum, determine the value of N that gives the maximum total weight of this fruit that would be produced per 100 m^2 for $10 \leq N \leq 25$.

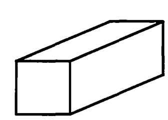
5. An open box is to be made by cutting squares from the corners of a rectangular sheet of card and folding up the resulting "flaps" to form the sides of the box.



   Use calculus ideas to determine the maximum capacity of such a box if the original card is (a) 25 cm by 40 cm, (b) 33 cm by 40 cm.

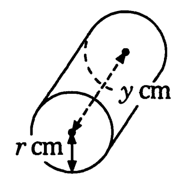
6. The cost, \$C, for the production of x units of a certain product is given by
   $$C = 0{\cdot}025x^2 + 2x + 1000, \quad x > 0.$$
   Use calculus to find the value of x for which the **average cost per unit** is a minimum, using the second derivative test to confirm a minimum, and find this minimum average cost.

7. 1000 cm^3 of metal is to be cast as a rectangular block with square ends. Use calculus to show that for the least surface area the rectangular block needs to be a cube.



8. A company makes and sells two products which we will call an A and a B.
Each A that the company makes gives a profit of \$5600.
Each B that the company makes gives a profit of \$200.
The company has the capacity to produce 20 As each month if it makes no Bs.
The company has the capacity to produce 400 Bs each month if it makes no As.
If the company chooses to make x As in a month, $0 \le x \le 20$, then it can only produce $(400 - x^2)$ Bs.
Use calculus and the second derivative test to determine how many of each the company should make each month to maximise its profit.
What would this maximum profit be?

9. The diagram shows a cylindrical container with end radius r cm and length y cm.
If the total of the circumference of a circular end and the length of the cylinder is to be 120 cm, find the values of r and y that will maximise the capacity of the container.



10. Fencing is to be used to construct an enclosed rectangular region. The area of the region is to be 8000 m^2. Fencing costing \$16 per metre is to be used for three sides and fencing costing \$24 per metre used for the fourth side.
Find the dimensions of the rectangle that will minimise the cost of fencing, and find this minimum cost.

11. The probability of a patient recovering from a particular disease is given by

$$\frac{15x}{64 + x^2} \quad \text{for } 0 \le x \le 20$$

where x is the number of units of drug X that is administered.
Use calculus to determine the value of x that gives the greatest probability of recovery, viewing the function on a graphic calculator to confirm that your value of x does indeed give a maximum, and determine this maximum.

12. Scientists monitor the spread of a particular disease through a population of native animals. They notice that whilst initially the proportion who have the disease increases, some of the population do not get the disease, possibly due to some of the animals carrying a natural immunity. They also find that with treatment the proportion of the population with the disease eventually decreases.
In an attempt to model P, the proportion of the animals with the disease, t months after it is thought to have started, the following rule is suggested:

$$P = \frac{18t}{t^2+5t+100}.$$

For this suggested rule, use calculus to determine the maximum value of P, the value of t for which it occurs, and use either the sign test or the second derivative test (whichever you deem most appropriate) to confirm that the P value is indeed a maximum.

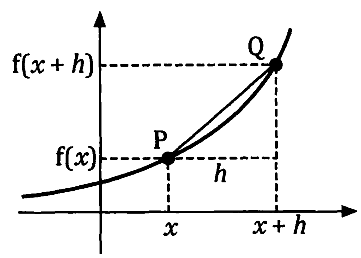
13. The cost, \$C, for the production of x units of a certain product is given by
$$C = (x + 10)^3, \quad x > 0.$$
Find the value of x for which the **average cost per unit** is a minimum and find this minimum average cost.

14. Vehicle A is 25 km due east of vehicle B.
A moves due west at 60 km/h and B moves due north at 80 km/h.
Find an expression for the distance of separation for the two vehicles t hours later.
After how many minutes is this separation distance a minimum and what is this minimum distance?

15. Performing any differentiation with a calculator that can determine gradient functions, show that for an isosceles triangle of fixed perimeter to have a maximum area the triangle needs to be equilateral.

Small changes.

The preliminary work reminded us of the fact that differentiation is based on the idea that for the curve on the right:



$$\text{Gradient at P} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Writing δx in place of h, the small increase, or increment, in the x coordinate,
and δy in place of $f(x + h) - f(x)$, the small increment in the y coordinate,

we arrived at

$$\text{Gradient function} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}. \qquad \text{We wrote this as } \frac{dy}{dx}.$$

Thus if δx, the change in x, is small then

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

Hence δy, the small change in y, caused by δx, the small change in x, can be approximately determined using

$$\delta y \approx \frac{dy}{dx}\,\delta x$$

In this way differentiation allows us to determine an approximation for the small change in one variable given the small change in another, related, variable.

If δx, the change in x, is small then	$\dfrac{dy}{dx} \approx \dfrac{\delta y}{\delta x}$
and so	$\delta y \approx \dfrac{dy}{dx}\,\delta x$

Note: It was mentioned in the Preliminary work section that the symbol δ is a Greek letter pronounced *delta.* The capital of this letter is written Δ and in this calculus context δx, the small change in x, is sometimes written as Δx.

Example 9

If $y = 3x^3 + 2x - 1$ use differentiation to find the approximate change in y when x changes from 2 to 2·01.

If $y = 3x^3 + 2x - 1$ then $\dfrac{dy}{dx} = 9x^2 + 2$

Thus for δx a small change in x, $\dfrac{\delta y}{\delta x} \approx 9x^2 + 2$

In this case $x = 2$ and $\delta x = 0{\cdot}01$, thus $\delta y \approx (9(2)^2 + 2)\,(0{\cdot}01) = 0{\cdot}38$

When x changes from 2 to 2·01 the change in y is approximately 0·38.

The reader should compare this approximate value for the change in y with the exact answer given by $(3(2{\cdot}01)^3 + 2(2{\cdot}01) - 1) - (3(2)^3 + 2(2) - 1)$.

Example 10

Find the approximate change in the area of a square when expansion causes the sides to increase from 20 cm to 20·25 cm.

The area, A, of a square is related to the side length, x, by the rule $A = x^2$.

If $A = x^2$ then $\dfrac{dA}{dx} = 2x$

Thus for δx a small change in x, $\dfrac{\delta A}{\delta x} \approx 2x$

In this case $x = 20$ and $\delta x = 0{\cdot}25$, thus
$$\delta A \approx 2(20)(0{\cdot}25) = 10$$

When the side length changes from 20 cm to 20·25 cm the approximate change in the area is 10 cm^2.

Again the reader should compare this approximate value for the change in area with that of $(20{\cdot}25)^2 - (20)^2$.

Example 11

Find, correct to four decimal places, the radius of a sphere of volume 1000 cm^3.
Use calculus to determine the approximate change necessary in the radius of the sphere to cause the volume to change from 1000 cm^3 to 1010 cm^3.

The volume, V, of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

If the volume is 1000 cm^3 then $1000 = \frac{4}{3}\pi r^3$.

Thus $r^3 = \frac{3000}{4\pi}$

Giving $r = 6{\cdot}2035$ (to 4 d.p.).

With $V = \frac{4}{3}\pi r^3$ then $\frac{\mathrm{d}V}{dr} = 4\pi r^2$

For δV a small change in V, $\frac{\delta V}{\delta r} \approx 4\pi r^2$

$\therefore \quad \frac{\delta r}{\delta V} \approx \frac{1}{4\pi r^2}$

In this case $r \approx 6{\cdot}2035$ and $\delta V = 10$, thus

$$\delta r \approx \frac{10}{4\pi(6{\cdot}2035)^2} \approx 0{\cdot}02$$

The radius of a sphere of volume 1000 cm^3 is 6·2035 cm, correct to four decimal places, and an increase of approximately 0·02 cm is required in the radius to increase the volume to 1010 cm^3.

Small percentage changes.

Example 12

If $V = 2x^3$ use differentiation to find the approximate percentage change in V when x changes by 2%.

In this question we are given $\frac{\delta x}{x} = \frac{2}{100}$ and are required to find $\frac{\delta V}{V}$.

If $V = 2x^3$ then $\frac{\mathrm{d}V}{\mathrm{d}x} = 6x^2$.

Thus for δx a small change in x, $\frac{\delta V}{\delta x} \approx 6x^2$.

Therefore

$$\frac{\delta V}{V} \approx \frac{6x^2\delta x}{V} = \frac{6x^2\delta x}{2x^3} = 3\frac{\delta x}{x} = \frac{6}{100}.$$

When x changes by 2%, V changes by approximately 6%.

Marginal rates of change.

Suppose a firm produces x units of a particular commodity. There are three important functions of x that will interest the firm.

- The cost function, $C(x)$. The cost of producing the x units.
- The revenue function, $R(x)$. The income from selling the x units.
- The profit function, $P(x) = R(x) - C(x)$.

Considering the cost function, if the number of units produced increases by 1, i.e. $\delta x = 1$,

then $$\frac{\delta C}{1} \approx \frac{dC}{dx}.$$

In this way $\frac{dC}{dx}$, called the **marginal cost**, gives the approximate cost of producing one more unit at the stage of production that has just seen the x^{th} unit produced.

Similarly:

$\frac{dR}{dx}$, the **marginal revenue**, gives the approximate extra revenue brought in by the sale of one more item after the x^{th} item has been sold.

$\frac{dP}{dx}$, the **marginal profit**, gives the approximate extra profit produced by the sale of one more item after the x^{th} item has been sold.

Example 13

A manufacturing firm produces and subsequently sells x items of a certain product. The total cost of producing these x items is \$C, with C given by

$$C(x) = 6x + 10\sqrt{x} + 500.$$

Use differentiation to determine the approximate cost of producing one more item at the stage in the production when $x = 100$.

With $C(x) = 6x + 10\sqrt{x} + 500$ then $$\frac{dC}{dx} = 6 + \frac{5}{\sqrt{x}}.$$

Thus $$\frac{\delta C}{\delta x} \approx 6 + \frac{5}{\sqrt{x}}$$

With $\delta x = 1$ and $x = 100$ $$\delta C \approx 6 + \frac{5}{\sqrt{100}}$$
$$= 6{\cdot}5$$

It will cost approximately \$6·50 to produce one more item at the stage in the production when $x = 100$.

Exercise 2E

1. If $f(x) = x^2 + 4x$ use differentiation to find the approximate change in the value of the function when x changes from 4 to 4·02.
Compare your answer to $f(4{\cdot}02) - f(4)$.

2. If $f(x) = 2x^2 - 5x$ use differentiation to find the approximate change in the value of the function when x changes from 3 to 3·01.
Compare your answer to $f(3{\cdot}01) - f(3)$.

3. If $y = x^3 + 4$ use differentiation to find the approximate change in y when x changes from 1 to 1·05.

4. If $y = 2x^3 - 4x$ use differentiation to find the approximate change in y when x changes from 5 to 5·01.

5. If $y = t^3 + 3t^2 - 6t + 4$ use differentiation to find the approximate change in y when t changes from 2 to 2·01.

6. If $y = \dfrac{1}{t+1}$ use differentiation to find the approximate change in y when t changes from 4 to 4·1.

7. If $y = \sqrt{t}$ use differentiation to find the approximate change in y when t changes from 25 to 26.

8. If $y = 3x^2$ use differentiation to determine the approximate percentage change in y when x increases by 5%.

9. If $y = t^3$ use differentiation to determine the approximate percentage change in y when t increases by 2%.

10. Use differentiation to determine the approximate change in the area of a circle when the radius changes from 10 cm to 10·1 cm.

11. Find, correct to three decimal places, the radius of a circle of area 120 cm^2. Use differentiation to determine the approximate change in the radius when the area changes from 120 cm^2 to 121 cm^2.

12. The cost of producing n units of a particular product is given by
$$C = n^3 - 45n^2 + 800n + 1000 \text{ (in dollars).}$$
Use differentiation to determine the approximate cost of producing one more item at the stage in the production when $n = 20$.

13. The total revenue, $\$R$, from the sale of x items is given by
$$R = 25x - 0{\cdot}01x^2.$$
Use differentiation to determine the approximate revenue increase from the sale of one more item at the stage in the production when $x = 200$.

14. A temperature change causes the radius of a metal sphere to contract from 10 cm to 9·99 cm. Use differentiation to find the approximate change in the surface area.

15. Let us suppose that a person's body surface area, A, is related to the person's weight according to the rule $A = \mathrm{k}W^{0\cdot 4}$ for some constant k.
Use differentiation to find the approximate percentage gain in A when W increases by 2%.

16. Use differentiation to find the approximate increase in the surface area of a spherical soap bubble if its radius changes from 2·5 cm to 2·6 cm.

17. The profit, $\$P$, a company makes by the production and sale of x items of a certain product is given by $$P = 20x^2 - 4000 - \frac{x^3}{12}.$$
Use differentiation to determine the approximate profit increase from the sale of one more item at the stage in the production when $x = 100$.

18. A company has the task of producing spheres of volume 288π cm^3 $\pm$ 5 cm^3. Use differentiation to determine the radius of the spheres giving your answer in the form a cm $\pm$ b cm with b rounded to 3 decimal places.

19. An oil slick is approximately circular of radius r m and thickness 5 cm. Find an expression for the volume of oil in the slick in m^3.
Use differentiation to determine the approximate increase in the radius if a further 1 m^3 of oil leaks into the slick when the radius is 20 m, the thickness of the slick remaining at 5 cm.

20. A hollow metal cube has an exterior edge length of 10 cm.
Use differentiation to find the approximate volume of metal required to make the cube if the walls are 2 mm thick.

21. The time period T for a simple pendulum of length l is given by $T = 2\pi\frac{\sqrt{l}}{\sqrt{\mathrm{g}}}$ where g is a constant.
Find the percentage change in T when l changes by 6%.

Miscellaneous Exercise Two.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Express each of the following in the form $a\sqrt{2}$.

(a)	$\sqrt{8}$	(b)	$\sqrt{32}$	(c)	$\sqrt{50}$
(d)	$\sqrt{18}$	(e)	$\sqrt{98} + 3\sqrt{2}$	(f)	$\sqrt{200} - \sqrt{72}$
(g)	$\dfrac{4}{\sqrt{2}}$	(h)	$5\sqrt{2} - \dfrac{2}{\sqrt{2}}$	(i)	$\dfrac{20}{\sqrt{2}} - \sqrt{128}$

2. By writing x^5 as $(x^3)(x^2)$ differentiate $y = x^5$ using the product rule.

3. By writing x^5 as $\dfrac{x^7}{x^2}$ differentiate $y = x^5$ using the quotient rule.

Differentiate each of the following with respect to x.

4.	$10x^2$	5.	$5x$	6.	$2x + 1$
7.	$2x^3 - 3x^2$	8.	$5x^3 + 7x^2 - 6x + 1$	9.	-9
10.	$(2x - 1)(3x + 2)$	11.	$3x^2(2x - 1)$	12.	$(2x + 5)^2$
13.	$(2x - 1)^7$	14.	$\dfrac{5x + 1}{2x - 3}$	15.	$\dfrac{5x + 1}{3x^2 - 1}$

16. If $x = 5t^3 + 3t + 6$ find an expression for $\dfrac{d^2x}{dt^2}$ and evaluate this for $t = 3$.

17. If $x = 3t^2 - \dfrac{2}{t}$ find an expression for $\dfrac{d^2x}{dt^2}$ and evaluate this for $t = 2$.

18. If $x = (2t + 3)^5$ find an expression for $\dfrac{d^2x}{dt^2}$ and evaluate this for $t = 1$.

19. If $x = t^3 + 20t^2 - 500\sqrt{t}$ find an expression for $\dfrac{d^2x}{dt^2}$ and evaluate this for $t = 25$.

20. If $A = 3x^2y$ and $y = (25 - 2x)^5$ show that $\dfrac{dA}{dx} = 6x(25 - 2x)^4(25 - 7x)$.

    Hence determine the values of x for which A has stationary points and use either the second derivative test or the sign test, whichever you consider the more appropriate, to determine the nature of each.

21. Find the equation of the tangent to $y = (x - 1)(x^2 - x - 5)$ at the point $(-1, 6)$.

22. The sensitivity of a drug is given by $\dfrac{dR}{dq}$ where R is the reaction to an injection of q units of the drug. If a particular drug is such that

$$R = \frac{q^2(400 - q)}{2}$$

find the sensitivity of the drug when (a) $q = 50$, and (b) $q = 100$.

23. (a) Use the quotient rule to determine $\dfrac{dy}{dx}$ for $y = \dfrac{x^2 + 2}{x - 1}$.

(b) Using your calculator if you wish, determine the exact coordinates of any stationary points on the curve

$$y = \frac{x^2 + 2}{x - 1}.$$

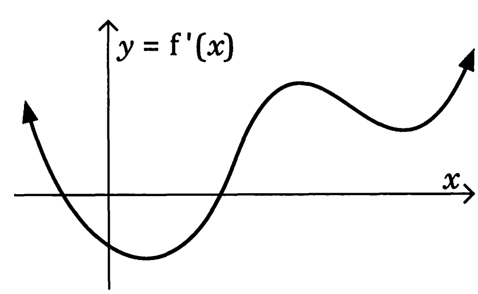
(c) By viewing the graph of $y = \dfrac{x^2 + 2}{x - 1}$ on a graphic calculator determine the nature of each stationary point.

24. The curve $y = ax^2 + b$ passes through the points A (3, 39) and B (−2, c).
The tangent to the curve at A is parallel to the line $y - 30x + 7 = 0$.
Find the values of the constants a, b and c and the equation of the tangent to the curve at B.

25. If $y = \dfrac{20}{x}$ use differentiation to determine the approximate percentage change in y when x increases by 2%.

26. The diagram on the right shows the graph of
$y = f'(x)$.

Sketch a possible graph for
(a) $y = f''(x)$
(b) $y = f(x)$



27. A function $y = f(x)$ is such that $f(3) = 1$
and $f''(3) = 0$.
Can we conclude that the graph of $y = f(x)$ has an inflection point at (3, 1)?
(Explain your answer.)